Graduate Institute of International and Development Studies
International Economics Department
Working Paper Series

Working Paper No. HEIDWP04-2019

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February 2020 version,
initial version: March 2019

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Abstract

We present a model of firm investment under uncertainty and partial irreversibility in which uncertainty is represented by a jump diffusion. This allows to represent both the continuous Gaussian volatility and the discontinuous uncertainty related to information arrival, sudden changes and large shocks. The model shows how both sources of uncertainty negatively impact the optimal investment and disinvestment policies, and how the presence of large negative jumps can drastically affect the firm's ability to recover. Our results show that the standard Gaussian framework consistently underestimates the negative effect of uncertainty on firm investment decisions. We test these predictions on a panel dataset of UK firms: we first structurally estimate the uncertainty parameters using multinomial maximum likelihood and differential evolution techniques and subsequently study their impact on firm investment rates, validating our model predictions.

Keywords: firm investment, uncertainty, jump diffusions, partial irreversibility, real options
JEL Codes: C61, C62, D21, D22, D8

1 Introduction

A firm's investment behaviour is receptive to the amount of uncertainty it face in market conditions and future prices. Contemporary studies use the real options approach introduced by Dixit and Pindyck (1994), which highlights the interaction of uncertainty over future returns to capital, the degree of irreversibility due to presence of adjustment costs and the option to delay investment. The typical literature on investment under uncertainty assumes the demand and productivity conditions faced by firms, mainly captured by their prices or returns, to follow a diffusion process. Therefore, in this framework, uncertainty is entirely conflated within the variance parameter of the Gaussian distribution, which is assumed to represent all sources of risk. Often, however, discontinuities and exogenous shocks ("jumps") are observed in the firm dynamics due to a variety of causes, such as information arrival, entry by a new competitor in a market with few firms, resignation of the CEO, significant changes in the operating and financial structure of

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the firm, regulatory changes introduced by the government, new innovations, et cetera. If such events are not forecastable, the times at which they arrive will be random, and we will have another measure of uncertainty in their arrival frequency. One must also not forget that the impact of such events on the environment the firm faces may be of different magnitudes: the sizes of the jumps, therefore, add another instrument to the list of measures of the uncertainty a firm faces.

The introduction of such discontinuities introduces non-Gaussianity in the firm dynamics by means of skewness and thicker tails, and escapes the restriction of having to represent all sources of risk by means of the variance of a Normal distribution. Because of these requirements, jump diffusion processes are an ideal candidate. A jump diffusion is part of a wide class of processes, the Lévy processes, which are continuous in probability and have stationary and independent increments. Lévy processes "allow" their sample paths to jump at specific points in time: in more technical terms, any Lévy process allows a version of itself that is continuous to the right and limiting to the left (the Brownian motion is a particular case of a Lévy process). Jump diffusions can be expressed as a linear combination of the Brownian motion and a Poisson jump process whose coefficients do not change in time, and whose jump sizes may be themselves random. Jump diffusions, therefore, represent a useful generalization of Itô diffusions that can represent well the uncertain environment faced by a firm.

In this paper we decompose the uncertainty faced by the firm into volatility and jumps in the demand conditions. Studying how both these sources of uncertainty affect the thresholds at which investment and disinvestment occur, and the sequential determination of the optimal policies, provides the main motivation for our model. Our model contributes to the literature by examining the firm's investment decisions under jump-diffusive dynamics with a real option framework and empirically investigating these effects. The model describes how firms constrained by partial irreversibility and non-convex adjustment costs react to uncertainty that is generated by the sum of two independent processes, a Brownian motion and a compound Poisson process with random jump magnitudes. Coherently with the literature, under these assumptions, firms face an *inaction area* in which it is not profitable to invest: we study how volatility, the frequency of jumps and their direction affect the size of this area and the consequent investment and disinvestment dynamics. The model has multiple implications. The first is that in the presence of jumps in the firm's business conditions, an increase in volatility pushes upwards the investment threshold much more than predicted by the traditional real option models, implying the delay in investment decisions are in fact underestimated. Second, an increase in the frequency of negative jumps always increases the size of the inaction region. Furthermore it prompts firms to disinvest quicker in order to protect themselves from a jump large enough to cause operations shutdown: in our framework this is possible even in the presence of partial irreversibility. Another implication of the model is that an increase in positive jump frequency has a positive effect on investment threshold only when the underlying volatility is low. For high levels of volatility this effect becomes negative. Lastly, with the inclusion of jumps, an increase in volatility and jump frequencies amplifies the inaction region for firms with low growth rate to a greater extent than for medium and high growing firms. We then test these results on a panel of 403 publicly listed U.K. firms by first structurally estimating the uncertainty parameters using multinomial maximum likelihood and differential evolution techniques. In the second stage, using system GMM we find evidence that both jumps and volatility reduce the impact effect of a positive demand shock on the firm's investment rate. Our empirical evidence corroborates our theoretical implications. The paper that is closest to ours is Boyarchenko (2004), however there are two key differences. First, we provide a richer market structure in the form of a profit function that encapsulates non-convex costs capturing a crucial aspect of investment behaviour i.e. it exhibits periods of inaction interrupted by investment spikes and is consistent with empirical literature seen in firm-level and plant-level data. Second, we explicitly model partial investment irreversibility. Our paper can be seen as building on Boyarchenko (2004), while providing a more economically intuitive framework and empirical evidence to validate the model results.

The paper is organised as follows. In Section 2 of the paper, we construct a model of a risk-neutral firm with an infinite horizon and characterize its optimal investment decisions with partial irreversibility and uncertainty. Section 3 presents the properties and the economic implications of the model's numerical solutions. Section 4 focuses on testing empirically the theoretical implications by developing an econometric specification and applying it to firm level investment data. Section 5 offers concluding remarks.

2 The Model

We consider a risk neutral firm in a continuous time economy, which chooses at every time interval the level of investment I_t^+ and disinvestment I_t^-. The pointwise operating profit of the firm is $\pi(K_t, X_t)$, and is assumed to be Cobb-Douglas with quasi-fixed costs:

$$\pi(X_t, K_t) = hX_t^{\gamma} K_t^{1-\gamma} - MK_t. \tag{1}$$

The profit function is determined by two state variables: K_t, the capital stock, and X_t, a variable that combines firm level productivity and demand terms into one index[1], which we will henceforth refer to as *business conditions* (using Bloom et al. (2007) terminology). The quasi-fixed costs represent production costs per unit of capital, and can also be understood as firm maintenance costs.[2] This profit function is consistent with a firm whose production technology exhibits diminishing returns to scale and is based on an underlying production function where a

[1]See the Appendix for the derivation of X and γ from a production function that includes both capital and labor.

[2]The scalars $h, M > 0$ and $0 < \gamma < 1$ determine the characteristics of the profit function similarly to what is assumed by Cooper (2006).

flexible factor of production (labor) is continuously optimized out.

The capital stock K_t is assumed to follow a deterministic law of motion, controlled by the firm which chooses at every time interval the level of investment $I^+ = dK^+$ and disinvestment $I^- = dK^-$:

$$dK_t = -\delta K_t dt + I^+ - I^- \tag{2}$$

where $\delta \in [0, 1]$ and $I^+, I^- \geq 0$. The firm incurs a total investment cost $C^i(X_t, K_t, I^i)$ which is made of non-convex costs with partial irreversibility, where $i = +, -$ indicates whether the firm is investing $(+)$ or disinvesting $(-)$. We have, with $0 < \alpha_i < 1$:

$$C^i(X_t, K_t, I^i) = \alpha_i \pi(X_t, K_t) + p_i I^i, \tag{3}$$

which implies that when a firm both invests (or disinvests) it loses a part of the profits as adjustment costs.

In our framework, p_+ is the price of buying capital (investment, I^+) and p_- the price of selling it (disinvestment, I^-). We have $p_- < 0 < p_+$ and $|p_-| < |p_+|$, and therefore partial irreversibility.

2.1 The Stochastic Process for Business Conditions

Business conditions, X_t, is assumed to follow a geometric Lévy process (a jump diffusion), continuous from the left, of the form

$$dX_t = \mu X_t dt + \sigma X_t dW_t + X_{t^-} \int_{\Omega} z \tilde{N}(dt, dz) \tag{4}$$

where the scalars $\mu \in \mathbb{R}, \sigma > 0$ are the drift and the diffusion parameters. W_t is the standard one-dimensional Brownian motion on a probability space endowed with the filtration $\mathcal{F}_t$, and $\tilde{N}(dt, dz)$ is the compensated Poisson measure on $\Omega = [-1, \infty)$ for a compound Poisson process of the form

$$\tilde{N}(dt, dz) = N(dt, dz) - \lambda F(dz) \tag{5}$$

where $\{N(t, z), t > 0\}$ is a Poisson process with intensity $\lambda \in \mathbb{R}^+$, and the the jump magnitudes are iid random variables with distribution $F(dz)$. This modeling may seem obscure but it has an intuitive interpretation: the jumps arrive with intensity λ and the magnitude is determined by the distribution $F(dz)$, for which so far we only assume $||F|| = F(-1, \infty) < \infty$ for boundedness. We work with jump diffusions to allow for only one jump to happen in the infinitesimal dt, a simplification that will be of key use in the estimation of the density parameters.[3]

[3] In technical terms, we assume a finite Lévy measure $\nu = \lambda F$ with support $\Omega = [-1, \infty)$. The fact that the integral in (4) is bounded at -1, as in Framstad et al. (2001) is to guarantee positivity of the state variable: because of the geometric nature of the process, the negative jump is bounded to $-X$, so it cannot jump an amount greater than the pre-jump level of the state variable, which is reasonable.

Note that we are still keeping the probability measure of the jump term completely general: a common baseline choice is to assume a normal distribution for the jumps, i.e. $F(z) = N(\mu_j, \sigma_j^2)$, which is the framework of the seminal work by Merton (1976). One, however, can easily model the jump measure as an asymmetric distribution with different frequency rates, and the dynamics of (4) would read

$$dX_t = \mu X_t dt + \sigma X_t dW_t + X_{t^-}\left(\int_{\Omega^-} z\tilde{N}_d(dt, dz) + \int_{\Omega^+} z\tilde{N}_u(dt, dz)\right) \tag{6}$$

where Ω^l are the respective domains of the positive and negative jump distributions. The Poisson measures $\tilde{N}_d(dt, dz), \tilde{N}_u(dt, dz)$ are

$$\begin{aligned}\tilde{N}_d(dt, dz) &= N_d(dt, dz) - \lambda_d F_d(dz)\\ \tilde{N}_u(dt, dz) &= N_u(dt, dz) - \lambda_u F_u(dz)\end{aligned}$$

where $N_r()$ has intensity λ^r, $r = \{\text{u,d}\}$, and the distributions F_r have the joint density

$$f(z) = \lambda_d \mathbb{1}_{z\leq 0} f_d(z) + \lambda_u \mathbb{1}_{z>0} f_u(z) \tag{7}$$

where f_u, f_d are the densities of the jump magnitudes. The asymmetry allows to model separately the influence of positive and negative jumps. Both finance and mathematics literatures present us with a plethora of possibilities: one of the most tractable yet still realistic models is the double exponential distribution, as in the celebrated model by Kou (2002), where both up and down jumps follow an exponential distribution with different rate parameters, i.e.

$$f(dz) = \lambda_D \eta_1 e^{z\eta_1} \mathbb{1}_{z\leq 0} + \lambda_U \eta_2 e^{-z\eta_2} \mathbb{1}_{z>0}. \tag{8}$$

where $1/\eta_1, 1/\eta_2$ are the means of the jump sizes. This requires to assume $1/\eta_1 > 1$ for boundedness of the jump, which implies the mean positive jump cannot exceed X_{t^-}, a reasonable restriction: together with the support of F bounded from under, allows to limit the overall magnitude of the jumps that otherwise would remain unbounded. The literature favors this distribution for having strong empirical backing, as well as being relatively simple to treat analytically, and represents well the skewed and leptokurtic features of market returns. Another distribution would be the Pareto-Beta distribution presented by Ramezani and Zeng (2007), where the positive jumps follow a Pareto distribution while the negative follow a Beta. Further possibilities are the double uniform distribution, studied by Hanson and J. Westman (2002), which is less driven by empirical evidence but allows to model the creation of genuine fat tails in the distribution, and would fit well the assumption of bounded negative jumps. Another possibility is the double Rayleigh distribution, as shown in Synowiec (2008). The common assumption in all of these distributions is that the up and down densities are additive. Each of the presented distributions can be used in all that follows, therefore we choose to keep the modeling general up to the simulations and the subsequent empirical analysis. The choice to include both positive and negative jumps allows us to include the impact of information arrival, as well as including discontinuities in the state variable that the purely Gaussian framework of the literature cannot incorporate.

2.2 Model Solution and Investment Dynamics

The objective function of the operating firm is therefore

$$J(K_t, X_t) = \mathbb{E}_t \int_s^{\infty} e^{-\rho s} \left[\pi(K_{t+s}, X_{t+s}) - \mathbb{1}_{I \neq 0} C^i(K_{t+s}, X_{t+s}, I^i)\right] ds \qquad (9)$$

and the firm has to choose optimal investment policies dI^+, dI^- in order to maximize (9) subject to the dynamic constraints (2) and (4). Since the cost function will activate even for very small capital adjustments, there will be an inaction area in the (X, K)-space for which the benefits of buying or selling capital will be overcome by the adjustment costs. In this inaction area, therefore, the value of the firm will be close to its maximum, and the marginal increase in value generated by an unitary capital purchase will not balance the costs it entails. The optimal investment policy therefore will be for the firm to stay idle until the profitability of investment will be "high enough" to justify the consequent adjustment costs. Once this profitability condition is realized, the firm will then determine the optimal amount of capital to be purchased, and the state variable K will then be brought back to the interval for which it is optimal for the firm to not invest. In technical terms, the problem becomes finding two double sequences v^+, v^- of the form

$$\begin{aligned} v^+ &= (\tau_1^+, \ldots, \tau_j^+, dK_1^+, \ldots, dK_j^+) \\ v^- &= (\tau_1^-, \ldots, \tau_j^-, dK_1^-, \ldots, dK_j^+) \end{aligned}$$

where the τ_j^i are $\mathcal{F}_t$-measurable hitting times and the dK_j^i are investment $(i = +)$ and disinvestment $(i = -)$. Given these two sequences, the (controlled) capital process is therefore

$$\begin{aligned} K_t^{(v^+)} &= K_t \qquad 0 \le t \le \min(\tau_1^+, \tau_1^-) \\ K_{\tau_1^+}^{(v^+)} &= K_t + dK^+ \\ K_{\tau_1^-}^{(v^-)} &= K_t - dK^- \\ dK_t &= -\delta K_t dt \qquad \tau_j^+ \le t \le \tau_{j+1}^+ \text{ and } \tau_j^- \le t \le \tau_{j+1}^- \end{aligned}$$

Let us define $\mathcal{S}$ the area in the (X, K)-space of inaction, for which it is not profitable for the firm to invest or disinvest. Define

$$\tau_S = \inf\{t \in (0, \infty); (X_t, K_t) \notin \mathcal{S}\},$$

assume the firm is given a set of admissible controls $\mathcal{V}$ and assume

$$\lim_{j \to \infty} \tau_j^i = \tau_S.$$

Then the firm's problem is to find a function $V(X, K)$ and v^{*+}, v^{*-} such that

$$V(X, K) = \sup_{v^i \in \mathcal{V}} J(X, K) \qquad i = +, -. \qquad (10)$$

This problem is equivalent to solving the Hamilton-Jacobi-Bellman (HJB) equation with zero optimal controls, as in the model by Abel and Eberly (1996): because of the presence of non-convex investment costs, within the firm's inaction area the two problems are equivalent. The HJB equation for such firm's problem is

$$\rho V^+(K_t, X_t)dt = \max_{I^+, I^-} \left[\pi(K_t, X_t) - \mathbb{1}_{I_t \neq 0} C(K_t, X_t, I^i) + \mathbb{E}_t dV^+(K_t, X_t) \right].$$

If the firm has the (costless) option to shut down, then the value of the firm is

$$V(K_t, X_t) = \max[V^+, 0]. \tag{11}$$

and if does not simply $V^+ = V$. The equation for our problem (or equivalently, the HJB for the inaction area), noticing that the jump increment of (4) is proportional to zX, can be obtained by the Ito-Lévy Lemma for a semimartingale and since the the compensated measure $dN_t - \lambda dt$ is a martingale we obtain

$$\begin{aligned} \rho V(K_t, X_t) &= \pi(K_t, X_t) - \delta K_t V_k + \mu X_t V_x + X_t^2 \frac{\sigma^2}{2} V_{xx} + \qquad (12) \\ &+ \lambda \int_\Omega [V(K_t, X_t(1+z)) - V(K_t, X_t) - z X_t V_x(K_t, X_t)] F(dz). \end{aligned}$$

which is a partial integro-differential equation (PIDE). In order to reduce the dimensionality of this equation we use the first degree homogeneity of the risk-neutral firm's well-behaving profit function, and consequently of the value function $V(.)$.

Coherently with the established literature, we express the value of the firm in the space of the scaled variable $s = X/K$ in order to reduce (12) to an ordinary equation, which is allowed by the homogeneity of the well-behaving profit function and is projected on the HJB equation by the assumption of risk-neutral firms. We want to show how does the variable X_t/K_t evolve in time. This is obtained in the following Proposition:

Proposition 1: *The scaled variable $s_t = X_t/K_t$ follows a geometric Lévy process given by*

$$ds_t = (\mu + \delta) s_t dt + \sigma s_t dW_t + s_{t^-} \int_\Omega z \tilde{N}(dt, dz) dt \tag{13}$$

Proof: See Appendix. ■

The variable K_t is deterministic so it's just a matter of scaling and it doesn't change the probabilistic properties of the framework: it allows us to study the HJB equation in the space of s_t instead of both X_t and K_t separately, avoiding the necessity to deal with a PIDE.

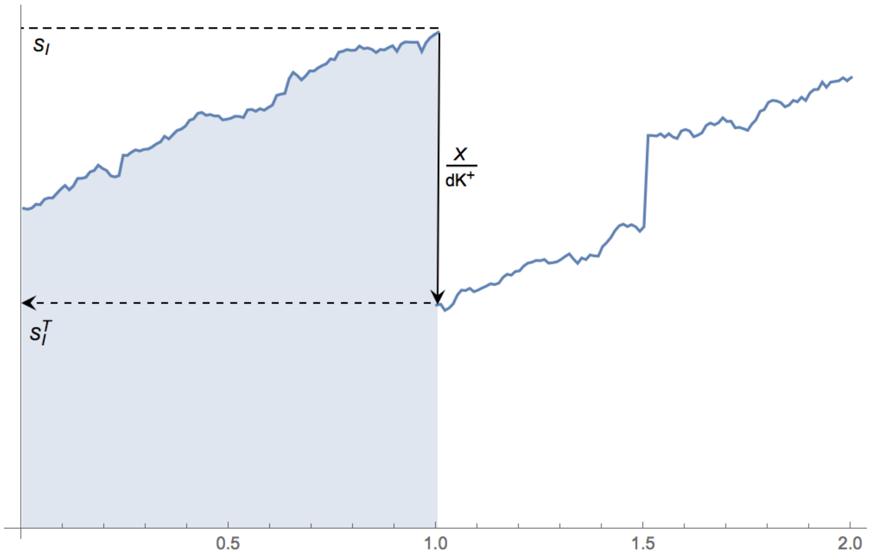


Figure 1: Simulation of investment dynamics. When s hits the upper threshold $s_I = X/K_t$, the optimal investment policy is to purchase an amount dK^+ of capital in order to shift the state variable to $s_I^T = X/(K_t + dK^+)$.

The inaction area translates into an interval on $\mathbb{R}$, $[s_I, s_D]$, for which it is optimal for the firm not to invest. When the state variable exceeds or equals this interval, i.e. $s_t \geq s_I$, the firm will purchase capital: dK will be positive and consequently $ds < 0$. This will move the state variable to a target point s_I^T inside the inaction interval. Vice versa for disinvestment: when the state variable s_t hits or exceeds the lower threshold s_D the firm will sell capital and reach the target s_D^T. The nature of the stochastic process that drives X allows for potential jumps of the variable on either side of the interval of inaction: intuitively, since the investment decisions will be based upon observation of the state variable s_t, its jumping dynamics allow for the state space of s to be outside the interval $[s_I, s_D]$. Figure 1 presents a simulation of the investment scenario.

Since $V(K_t, X_t) = K_t V(s_t)$ because of the zero-order homogeneity of its derivatives, using the form of the Lévy process that drives s_t and taking the derivative of $V(.)$ with respect to K_t, we define $q(s_t) = V_K(X_t, K_t)$ as the marginal product of capital; we can then write the new HJB equation for the region for which investment is zero in the space of s_t as

$$\begin{aligned}(\rho+\delta)q(s_t) &= (1-\gamma)as_t^{\gamma} + s_t(\mu+\delta)q'(s_t) + s_t^2\frac{\sigma^2}{2}q''(s_t) \\ &\quad +\lambda\int_{\Omega}[q(s_t(1+z)) - q(s_t) - s_t q'(s_t)z]F(dz). \qquad (14)\end{aligned}$$

We first note the presence of higher moments of the distribution of s in the solution of the problem, in the form of the higher moments of the jump distribution: the first probability integral can be expressed with as

$$\int_\Omega q(s_t + s_t z)F(dz) = \int_\Omega \sum_{n=0}^{\infty} \frac{q^{(n)}(s_t)}{n!}(s_t z)^n F(dz)$$

which allows us to write the following expansion:

$$\begin{aligned}
&\lambda \int_{\mathbb{R}} \Big[q(s_t(1+z)) - q(s_t) - s_t q'(s_t) z\Big] F(dz) = \\
&\lambda \left(\sum_{i=2}^{n} \frac{q^i(s_t)}{i!} \int_\Omega z^i F(dz) \right) + \mathcal{O}(z^{n+1})
\end{aligned} \tag{15}$$

where n is the highest moment of interest. This shows that for non-Gaussian, heavy-tailed distributions the solution of the firm's impulse control problem, and consequently the investment and disinvestment thresholds, depend on moments higher than mean and variance, due to the presence of skewness and kurtosis in the process that drives the business conditions.

We start by guessing that the function $q(s_t)$ is homogeneous of degree γ: if our guess is right, then we have a second-order ordinary differential equation instead of an integro-differential equation. The form of the solution for such an equation is known and is the sum of the homogeneous and complementary solutions, and it is an established result in the real options investment literature. We can then state the following proposition:

Proposition 2: *The firm's investment and disinvestment policies are determined by a set of five points $s_I, s_D, s_{IT}, s_{DT}, s_L$ which determine respectively the thresholds, the target levels and the shutdown point. These points characterize completely the impulse control of the value of the firm $\bar{V}(s_t) = K_t V(X_t, K_t)$ given by*

$$\bar{V}(s_t) = A s_t^\gamma - B + C_1 s_t^{\xi_1} + C_2 s_t^{\xi_2} \tag{16}$$

$$A = \frac{a}{\rho + \delta - \lambda\phi(z) - \gamma\left(\mu + \delta - \frac{\sigma^2}{2}\right) - \gamma^2 \frac{\sigma^2}{2}} \tag{17}$$

$$B = \frac{M}{\rho + \delta} \tag{18}$$

which is the sum of the value of the firm's asset in place and the options of both disinvesting and investing. The two exponents $\xi_1 < 0, \xi_2 > 0$ are given by

$$\begin{aligned}
\xi_{2,1} = \frac{1}{\sigma^2} \Bigg[&\left(\frac{\sigma^2}{2} - \mu - \delta + \lambda\phi_1(z) \right) \pm \\
&\pm \sqrt{\left(\mu + \delta - \lambda\phi_1(z) - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2 \left[\rho + \delta + \lambda\phi_2(z)\right]} \Bigg].
\end{aligned} \tag{19}$$

which imply that that the option of selling capital becomes less valuable as demand increases or as the capital stock decreases.The jump parts are given by the probability integrals

$$\begin{aligned}
\phi(z) &= \int_\Omega \Big[(1+z)^\gamma - 1 - \gamma z\Big] F(dz) && (20)\\
\phi_1(z) &= \gamma \int_\Omega zF(dz) && (21)\\
\phi_2(z) &= \int_\Omega [1-(1+z)^\gamma] F(dz). && (22)
\end{aligned}$$

The set of six positive constants $C_1, C_2, s_I, s_D, s_{IT}, s_{DT}$ are determined by the system of six nonlinear equations:

$$\begin{aligned}
\tilde{V}_I(s_I) &= \tilde{V}_I(s_{IT}) - \alpha^+ s_I^{1-\gamma} - p_1(s_{IT}^{-1} - s_I^{-1}) && (23)\\
\tilde{V}_D(s_D) &= \tilde{V}_D(s_{DT}) - \alpha^- s_D^{1-\gamma} - p_0(s_{DT}^{-1} - s_D^{-1}) && (24)\\
q(s_{IT}) &= p_1 && (25)\\
q(s_{DT}) &= p_0 && (26)\\
q(s_I) &= -\alpha^+(1-\gamma)s_I^\gamma + p_1 && (27)\\
q(s_D) &= -\alpha^-(1-\gamma)s_D^\gamma + p_0 && (28)
\end{aligned}$$

where $\tilde{V} = V(X,K)/X$ is the value function scaled by business conditions and $q(s) = V_K(X,K)$ is the marginal valuation of capital. Once the system is solved, the shutdown point will be given by

$$B = \frac{\tilde{A}}{1-\gamma} s_L^\gamma + \frac{C_1}{1-\xi_1} s_L^{\xi_1} + \frac{C_2}{1-\xi_2} s_L^{\xi_2}. \qquad (29)$$

Proof: See Appendix. ■

For all values of s_t for which $\bar{V} > 0$ and all feasible parameter sets,[4] then $K_t \bar{V}(s_t)$ is the value of the firm.[5]

[4] We have a regularity condition on the drift given by

$$\mu < \frac{1}{\gamma}(\rho + \delta - \lambda\phi(z)) - \delta + \frac{\sigma^2}{2}(1-\gamma).$$

[5] For asymmetric distributions, the above probability integrals will read

$$\begin{aligned}
\lambda\phi_1(z) &\rightarrow \gamma \left(\lambda_D \int_{\Omega^-} zF_d(dz) + \lambda_U \int_{\Omega^+} zF_u(dz)\right)\\
\lambda\phi_2(z) &\rightarrow \lambda_D \int_{\Omega^-} [1-(1+z)^\gamma] zF_d(dz) + \lambda_U \left(1 - \int_{\Omega^+} (1+z)^\gamma] zF_u(dz)\right)
\end{aligned}$$

and $\lambda\phi(z)$ will be the sum of the previous two expressions, where Ω_i are the respective domains of the up and down jump measures.

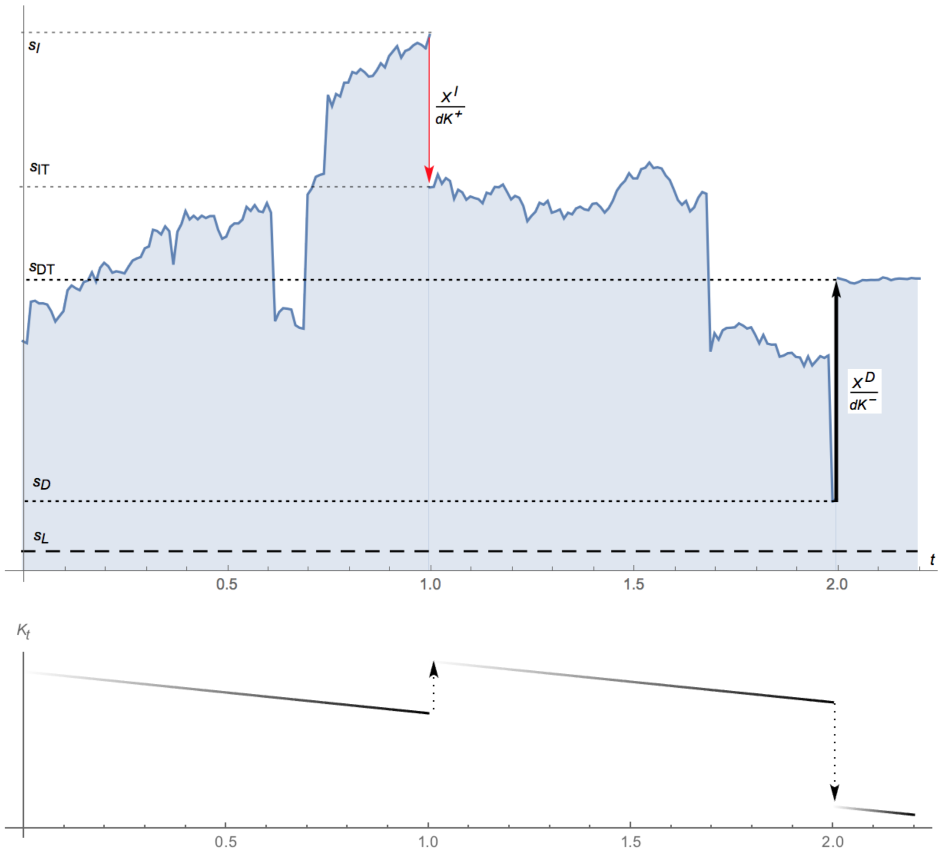


Figure 2: Upper panel: Simulation of investment dynamics with both investment and disinvestment. Parameters: $\mu = 0.1, \sigma = 0.1, \rho = 0.01, \delta = 0.05, \lambda_U = 0.5, \lambda_D = 0.5, \eta_1 = 2, \eta_2 = 5.$. The dashed line at s_L is the absorbing barrier of firm shutdown. Lower panel: Capital dynamics with depreciation

The optimal investment policies are therefore obtained in the following way: whenever the demand reaches the higher threshold X^I and K, investment is triggered to the target level $K_I > K$ and the firm incurs in the adjustment costs. In other words, the threshold is the point in the space (X, K) at which the increase in firm value that investment generates is exactly balanced by its costs, and the investment in order to be optimal will have to translate in a target level (X_I, K_I) that generates as much value, net of the costs that investment triggers. This translates in the following value matching condition:

$$V(X_I, K_t) = V(X_I, K_I) - \alpha^+ X_I^\gamma K^{1-\gamma} - p_1(K_I - K),$$

which means that the value of the firm at the target level of capital K_I for the same level of x_I must match the value of the firm at the level that triggered investment, minus the borne adjustment costs and the cost of purchasing capital. Since this condition involves only the threshold level X_I for the demand variable, we can rewrite it in scaled form. Since $\tilde{V}_I = V/X_I$, we obtain

$$\tilde{V}_I(s_I) = \tilde{V}_I(s_{IT}) - \alpha^+ s_I^{1-\gamma} - p_1(s_{IT}^{-1} - s_I^{-1}),$$

where $s_I = X_I/K_t$ is the value of the scaled demand X/K that triggers investment once reached, and s_{IT} is the target level to which the firm resets the state variable s by means of a purchase of capital $K_I - K_t$ for any initial level of capital K_t. The other two conditions are for optimality of the target point and for continuity of the derivatives, and follow from the zero-order homogeneity of the derivative of the value function:

$$\begin{aligned} q(s_{IT}) &= p_1 \\ q(s_I) &= -\alpha^+(1-\gamma)s_I^\gamma + p_1 \end{aligned}$$

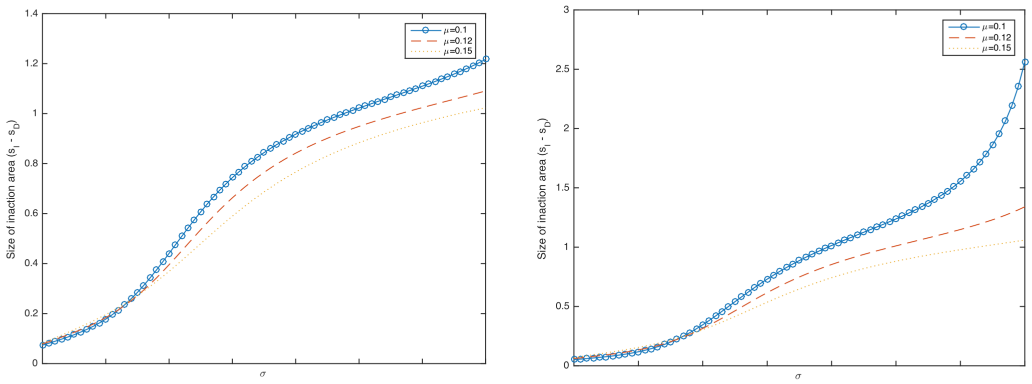
Note that (25) corresponds to the neoclassical condition for the investment threshold, which states that the target point for the scaled demand s_I^T is required to reflect the fact that in such a point the marginal valuation of capital must equal the cost of capital, and the ratio between the marginal productivity of capital q and the price of capital is unitary. Condition (27) has a similar interpretation: at the point in the (X, K)-space that triggers investment, it must also be that the marginal revenue of installed capital equals the price of buying capital, but since such point bounds the inaction area, it must also include the marginal adjustment costs (foregoing of part of the profits). We justified the boundary conditions by means of an intuitive economic jist: for the more rigorous arguments used for the mathematical derivation of the boundary conditions for impulse control of jump diffusions we refer to Øksendal and Sulem (2004). The same reasoning applies for disinvestment: once the high demand threshold X_D is reached for any level of capital, reflected in the threshold s_D, then the firm will sell capital to the target level K_D, bringing the state variable to its target level s_{DT}. This is reflected in the value matching and smooth pasting conditions

$$\begin{aligned} \tilde{V}_D(s_D) &= \tilde{V}_D(s_{DT}) - \alpha^- s_D^{1-\gamma} - p_0(s_{DT}^{-1} - s_D^{-1}) \\ q(s_{DT}) &= p_0 \\ q(s_D) &= -\alpha^-(1-\gamma)s_D^\gamma + p_0. \end{aligned}$$

where similarly to the investment boundary we define $\tilde{V}_D = V/X_D$. Note that $(s_D^T)^{-1} - s_D^{-1}$ is negative. Using the form of q and V given by (39) and (C), we are left with a nonlinear system of six equations in six variables which allows to identify the constants C_1 and C_2, the value of the investment and disinvestment options, and the thresholds and target levels s_I, s_I^T, s_D, s_D^T. If the process that drives X (and therefore s) was a jumpless diffusion, the state variable s would only oscillate between the thresholds s_I and s_D. Because of the possibility of a large enough negative jump is nonzero, there will be a point s_L below the disinvestment threshold which will act as an attracting barrier, for which the total value of the assets in place and the real options equals the costs of operating. The firm will then shut down operations: this point is given by the condition

$$\frac{\tilde{A}}{1-\gamma}s_L^\gamma + \frac{C_1}{1-\xi_1}s_L^{\xi_1} + \frac{C_2}{1-\xi_2}s_L^{\xi_2} = B.$$

Note that with a Gaussian diffusion driving the demand this point would be never



(a) $\lambda = 0$ (b) $\lambda = 5$

Figure 3: Increase in inaction area for an increase in $\sigma \in [0.05, 0.3]$, for different levels of drift μ and a Gaussian jump diffusion with constant frequency λ and $\mu_J = 0$.

reached, since allowing partial irreversibility allows the firm to sell capital once demand is low enough and the lower threshold s_D is hit, and the purely diffusive business conditions by their continuous nature cannot allow the threshold to be jumped. A simulation of the full firm dynamics can be seen in Figure 2, where the firm is simulated with the double exponential jump diffusion given by (8). The dashed line at s_L is the absorbing barrier of firm shutdown. When s hits the upper threshold $s_I = X/K_t$ at the first upper passage time τ_{1I}, the optimal investment policy is to purchase an amount dK^+ of capital in order to shift the state variable to $s_I^T = X/(K_t + dK^+)$, and when s hits the lower threshold at the first lower passage time τ_{1D}, corresponding to "bad" business conditions, then the optimal policy is to sell an amount dK^- of capital shifting the state variable back to s_D^T. Simultaneously the capital stock K_t depreciates "naturally", and its decrease is plotted on the lower panel of Figure 2. This also implies that investment is necessary even if the business conditions were to be constant, i.e. $dX_t = 0$: the used capital (together with its maintenance cost M) will decrease K_t and therefore the state variable s will be naturally pushed towards the upper investment boundary, at which the firm will replace the exhausted capital stock. Note that in this case the first upper passage time τ_{1K} is deterministic.

3 Numerical Solutions and their Implications

The system determined by (23) - (29) is not tractable analytically, and we now proceed to solve it numerically in order to study the impact of different components of uncertainty on the investment and disinvestment thresholds. This will allow us to show the model's stylized predictions on investment dynamics which we will then test empirically. We start by noticing that the equation (29) that determines the lower threshold s_L is decoupled, and that given s_{IT}, s_{DT}, s_I, s_D the system is linear in C_1, C_2. Given that we will solve the system on a grid of fixed parameters and then let different parameters of interest vary, this helps greatly in terms of

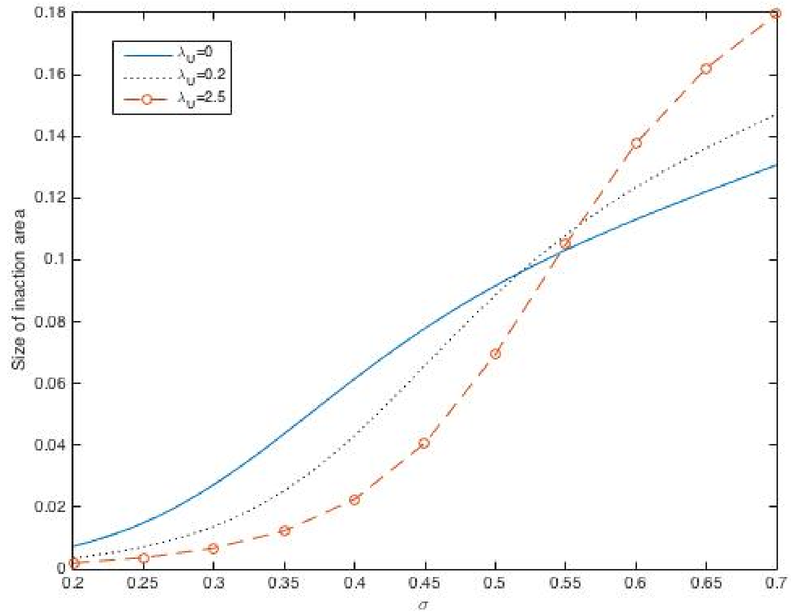


Figure 4: Impact of an increase in σ, for different levels of positive jump frequencies λ_U.

computation time. In all what follows we will assume the parameter values $\delta = 0.01, \rho = 0.01, \eta_1 = 0.5, \eta_2 = 0.5, \alpha^+ = 0.01, \alpha^- = 0.02, \gamma = 0.85, p_0 = 0.81, p_1 = 1, F = 0.05$.

We first show in Figure 3 how the size of the inaction area reacts to an increase in σ, for different levels of drift (μ), for both when jumps are present and not present. Consistent with the results in literature, in panel (a) we see in the absence of any jumps ($\lambda = 0$) i.e. a Gaussian diffusion process, an increase in volatility increases the inaction area of firms inversely related to their growth rates. Panel (b) of Figure 3 plots the size of the inaction area for a Gaussian jump diffusion, where the jump sizes are normally distributed. The figure shows that the presence of jumps increases the inaction area much further, and has an even bigger impact as volatility increases. This implies that regardless of the direction of jumps and/or if their mean (μ_J) is zero (i.e. negative and positive jumps are equally likely), the effect of jumps on the firms' inaction region continues to be increasing. What is especially worth noting here is that in the presence of high volatility, jumps drastically amplify the gap between low growing and medium to high growing firms.

We now show results obtained using a double exponential distribution with different jump frequencies λ_U, λ_D in order to study the effect of the frequency of positive and negative jumps separately. We first show that as the frequency of upward jumps λ_U increases, its impact on the inaction area depends on the level of "continuous" component of uncertainty σ: Figure 4 shows that an increase in λ_U for lesser levels of σ has a positive effect on investment threshold, since the inaction

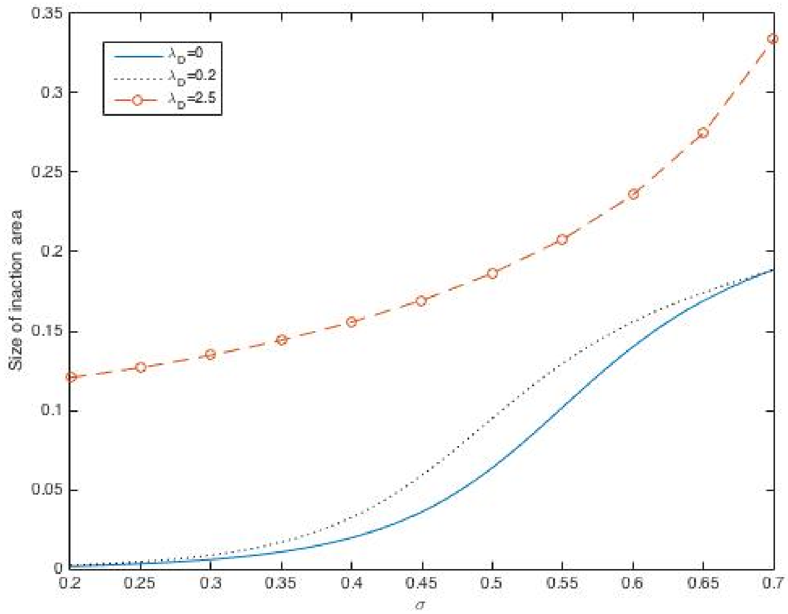


Figure 5: Impact of an increase in σ, for different negative jump frequencies λ_D.

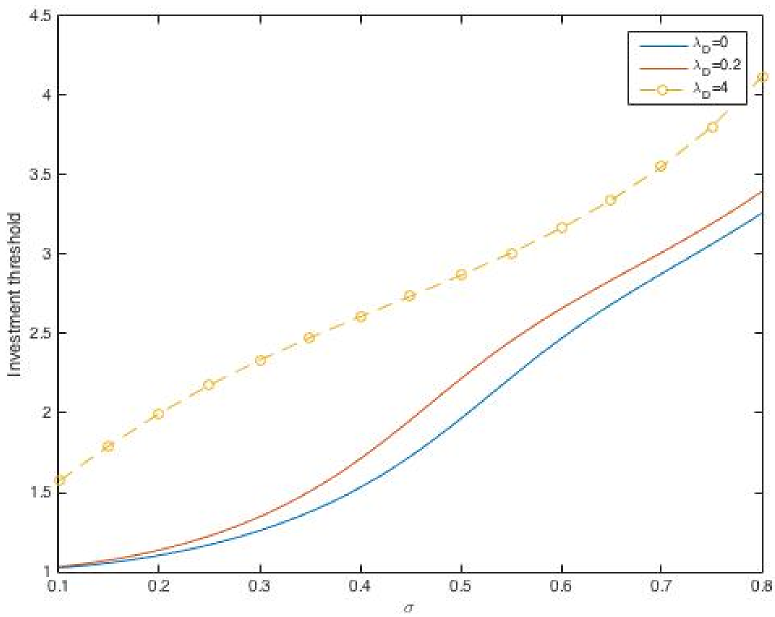


Figure 6: Impact on the investment threshold of an increase in σ, for different negative jump frequencies λ_D.

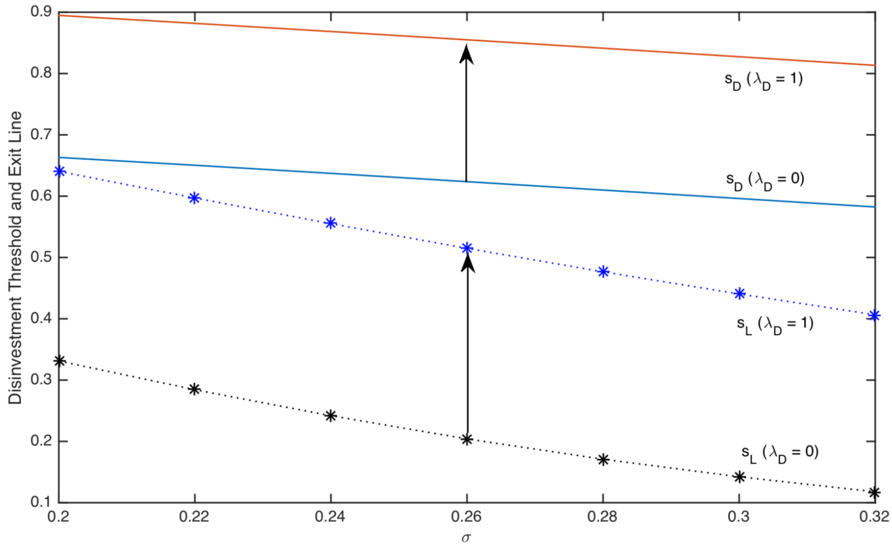


Figure 7: Impact on the disinvestment threshold and the exit line of an increase in σ, for different negative jump frequencies λ_D.

area decreases: this implies that if continuous uncertainty is low, and the overall environment is relatively stable, then up jumps will benefit firm investment, but if σ starts to increase above a certain level then up jumps will be considered simply as extra uncertainty, regardless of the fact that they could be beneficial. Next we explore how an increase in frequency of negative jumps λ_D pushes further away the investment thresholds: Figure 5 shows if the frequency of negative jumps is very small, then it will increase the inaction area until it is balanced by a dominating continuous uncertainty parameter σ, whereas if λ_D is large enough, the increase in the inaction area will be unequivocal. We further show the negative impact on investment by means of Figure 6, which shows the effect of σ and λ_D on the upper investment threshold s_I. The figure shows that again the magnitude of σ affects the impact of the jumps depending on the magnitude of the jump frequencies, always increasing the investment inaction area. Finally, in figure 7 we study the impact of both continuous and discontinuous uncertainty on the disinvestment threshold and on the exit line, i.e. the level of scaled business condition s_L at which the firm shuts down operations: an increase in σ pushes the threshold s_D down, consistently with the established literature results. The novelty here lies in the impact of an increase in the frequency of negative jumps, λ_D: the disinvestment threshold increases, as can be seen in Figure 7. The increase in s_D is inferior to the increase in the investment threshold s_I, implying an increase in inaction area. This can be interpreted by means of the division of uncertainty in continuous and discontinuous: the possibility of a negative jump large enough to project the firm's business condition below the exit line s_L increases the firm's propensity to disinvest and sell capital, as though to "ensure" enough distance from the absorbing barrier of the exit line s_L. This phenomenon is further strengthened by the increase in s_L (the exit point) which is greater than the increase in s_D, meaning that the disinvestment point and exit point become closer as the frequency of negative jumps increases. This phenomenon by construction cannot be described by a stochastic process that admits only continuous paths, and allows to show that the different forms that uncertainty can take can have radically different effects on the firm's investment decisions.

4 Empirics

Using the empirical strategy of Bloom et al. (2007) and Bond et al. (1999), we test the implications of our model on firm-level data. To investigate how volatility in the business conditions influences the firm's investment decisions, Bloom et al. (2007) uses an interaction term between demand shock and uncertainty to establish that sales growth or a demand shock has a weaker impact effect on current investment for firms facing higher levels of uncertainty[6]. Consistent with this we decompose our uncertainty in two components and use a similar interaction term to delineate the effects of jumps and volatility. We use an unbalanced panel of 403 publicly traded U.K. firms between 2003 and 2017[7]. Our empirical strategy consists of a two stage estimation: in the first stage we structurally estimate the uncertainty parameters for each firm in our sample. In the second stage, using these estimated parameters we conduct a firm level panel analysis using fixed effects model and system GMM, to see how a firm's responsiveness to investment is influenced by these parameters. We use Datastream to obtain the firm stock prices and Worldscope company accounts' annual data for firm fundamentals. Further information on the data is provided in the appendix.

A key caveat of our model is the assumption of constant volatility and jump frequency underlying the firm's business conditions. Although this is not a very practical assumption it is important in three ways: a) it allows our jump diffusion model to be analytically tractable, b) this makes the first stage structural estimation of the uncertainty parameters less computationally intensive and c) most importantly, although constant volatility is not a realistic feature of the data, it is a good starting point to disentangle the volatility and jump component of uncertainty and study their effects on firm investment behaviour. Consistently with the model assumptions, in this section we estimate time-invariant uncertainty parameters, such as volatility, jump frequencies and the parameters related to the jump size distribution. Time-invariance of the parameters does not affect the second stage panel analysis, since our principal interest is the interaction between real sales growth (demand shock) and the uncertainty components.

[6]This is a less ambiguous empirical result, robust to aggregation across investment decisions in multiple capital goods. Due to the presence of non-convex costs of adjustment, such as partial irreversibility, the impact of uncertainty on the average level of the capital stock in the long run is found to be ambiguous (Abel and Eberly (1999), Caballero (1999), Bloom (2000)), as both investment and disinvestment are deterred by real option effects

[7]As per convention, we drop firms classified under the following sectors: Banks, Financial Services, Life and Non-Life Insurance and Investment Trusts.

4.1 Stage One Estimation

The first stage of the estimation involves estimating the structural parameters of the stochastic process given by (4). If we could observe the true underlying business conditions (demand and productivity) of these firms or if we had access to output price data at a sufficiently disaggregated level, we could estimate directly the parameters from these time series. Unfortunately there is no such data availability, and therefore we use the high-frequency stock market returns data of the firms in our sample as a proxy for business conditions. This is in line with Leahy and Whited (1995) and Bloom et al. (2007), who state that the advantage of using asset returns is that it captures the effects of any aspect of a firm's environment that investors deem important, thus essentially capturing to an extent the business conditions.

For the estimation of the parameters, we restrict our attention to jump diffusions, so the jump measure is a not "pure" Lévy process. We therefore estimate a model of the form

$$dS_t/S_t = \mu dt + \sigma dW_t + z dN_t \tag{30}$$

and $S_0 = s_0$, where $\mu \in \mathbb{R}, \sigma > 0$ are scalars, W_t is the standard Brownian motion and z is a random jump magnitude. The discontinuous jump process is a compound Poisson process of the form

$$\int_{t_1}^{t_2} F(z) dN_t = \sum_{i=1}^{N_{t_2 - t_1}} z_i$$

where z_i is a sequence of i.i.d. random variables, and N_t is a λ-parameterized Poisson process. Since we bounded the negative jumps to -1, we choose a variable $\tilde{z} = \ln(1+z)$ and we satisfy the constraint. Solving the stochastic differential equation (30) one obtains immediately

$$S_t = s_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma W_t + \sum_{i=1}^{N_t} \tilde{z}_i\right]$$

and integrating over $t, t+\Delta t$ one gets

$$\Delta \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \Delta W_t + \tilde{z}\Delta N_t, \tag{31}$$

where $\Delta x_t = x_{t+dt} - x_t$ and Δt is chosen according to the frequency of the data[8]. We need now to choose the distribution of the random jump magnitude: for a first baseline estimation, we fit a Gaussian jump diffusion (GJD) on the firms' daily returns, in which the jump sizes are normally distributed, i.e. $\tilde{z} \sim N(\mu_J, \sigma_J^2)$. For the main estimation we first address one of the main concerns in using high-frequency stock market returns, which is that they may reflect noise unrelated to

[8] In all what follows we will deal with daily stock price data, and therefore we will choose $\Delta t = 1/252$ in order to obtain annualised estimates for the subsequent second part of the estimation.

firm fundamentals (Bloom et al. (2007); Gilchrist et al. (2014)). We address this issue by regressing with ordinary least squares (OLS) the firm daily returns on the benchmark FTSE All-Share Index and extracting the residuals, thus purging the systematic risk and focusing on the idiosyncratic component[9]. On these residuals, we fit an asymmetric double exponential jump diffusion (DEJD) of the form given by (8).

It is known that the density of log-returns given by (31) with the assumption of Gaussian jump sizes is an infinite mixture of Gaussian distributions, as shown for example by Honoré (1998), and for the DEJD model an even larger mixture as shown by Ramezani and Zeng (2007). Since we need to fit the models on daily observations ranging from 2003 to 2017, on an average of approximately 5000 observations per time series, for each of the 400+ firms, computation time for estimating such mixtures (even for low-degree truncations) can be an issue. Furthermore, for the DEJD model estimating such mixture has a variety of convergence issues. We therefore resort to a simplification first introduced by Ball and Torous (1983), which states that if the jumps do not happen too frequently, then in a small enough Δt only one jump can occur. Then ΔN_t can be approximated with a Bernoulli variable and the density of ΔS_t is given by

$$f_{\Delta S_t}(x;\theta) \sim (1-\lambda\Delta t)f_W(x) + \lambda\Delta t f_{W*Z}(x) + O(\Delta t^2)$$

where f_W is the density of the diffusive part of (30) f_{W*Z} is the convolution $\int_{-\infty}^{\infty} f_W(x)f_{W*Z}(x-y)dx$. For the Gaussian model, the convolution of two Gaussians is again Gaussian; f_W is a Gaussian density with mean $(\mu - 1/2\sigma^2)\Delta t$ and variance $\sigma^2\Delta t$, and f_{W*Z} is a Gaussian density with mean $(\mu - 1/2\sigma^2)\Delta t + \mu_J$ and variance $\sigma^2\Delta t + \sigma_J^2$ and the parameter vector $\theta = (\mu, \sigma, \mu_J, \sigma_J)$. For the DEJD, it can be shown that the density reads

$$\begin{aligned} f_{\Delta S_t}(x;\theta) &= \frac{1-(\lambda_U+\lambda_D)\Delta t}{\sqrt{\sigma^2\Delta t}}\phi(a_D) + \\ &+ \lambda_D\eta_1 e^{\frac{1}{2}\eta_1^2\sigma^2\Delta t} e^{(x-(\mu-\frac{1}{2}\sigma^2)\Delta t)\eta_1}\Phi(a_W) \\ &+ \lambda_U\eta_2 e^{\frac{1}{2}\eta_2^2\sigma^2\Delta t} e^{-(x-(\mu-\frac{1}{2}\sigma^2)\Delta t)\eta_2}\Phi(a_U) \end{aligned}$$

where $\theta = (\lambda_U, \lambda_D, \mu, \sigma, \eta_1, \eta_2)$, ϕ and Φ are the density and distribution functions of $N(0,1)$ and their arguments are given by

$$\begin{aligned} a_W &= \frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sqrt{\sigma^2\Delta t}} \\ a_D &= -\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t + \eta_1\sigma^2\Delta t}{\sqrt{\sigma^2\Delta t}} \\ a_U &= \frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t - \eta_2\sigma^2\Delta t}{\sqrt{\sigma^2\Delta t}}. \end{aligned}$$

[9]This scenario is analogous to the Capital Asset Pricing Model (CAPM) and much research in finance has focused on idiosyncratic volatility. See Campbell et al. (2001) and Bekaert et al. (2012)

In such mixture settings, there may exist several local minima. The estimation of the model is therefore a difficult task and we use two estimation techniques that help circumvent the issue of multiple stable points: *multinomial maximum likelihood* and *differential evolution*. For the first, we implemented the following procedure, as presented by Hanson et al. (2004) and for whose robustness and consistency properties we refer to the original paper. We first sort the n observations of the data vector in b bins of equal size $B_i = [\xi_{i-0.5\Delta b}, \xi_{i+0.5\Delta b}]$. We choose for bin size

$$\Delta b = \frac{(x_{max} - x_{min})\sqrt{n}}{q_{0.75} - q_{0.25}}$$

where q_a is the a-th percentile of the data series. We then calculate the theoretical jump diffusion frequency for each bin as

$$f_i^{th}(\theta) = n \int_{B_i} f_{\Delta S_t}(x;\theta)dx,$$

and obtain the multinomial likelihood to be minimized as

$$MLL(\theta) = -\sum_{i=1}^{b} f_i^{emp} \ln f_i^{th}(\theta) \tag{32}$$

where f_i^{emp} is the observed frequency of the data for each i-th bin. The minimization of (32) yields the MLL estimator $\hat{\theta}$. Note that $\mathbb{E}f_i^{emp} = f_i^{th}(\theta)$, therefore we have that the mean objective is

$$\mathbb{E}\big[MLL(\theta)\big] = -\sum_{i=1}^{b} f_i^{th}(\theta) \ln f_i^{th}(\theta)$$

which means that the mean objective is the *entropy of the information in each bin*. Particular care must be taken while estimating the DEJD to bound the parameter space for the integrals not to explode: we avoid this issue by bounding below the parameters σ and σ_J to 10^{-5}, as well as appropriately bounding the rest of the parameters to "reasonable" intervals. Lastly, we obtain standard errors from the diagonal of the information matrix of $MLL(\theta)$. Furthermore, we set to zero each parameter for which a zero null hypothesis is not rejected at 95% confidence: this in our sample only happened within the jump frequency parameters[10].

The second estimation technique involves using differential evolution in order to minimize the log-likelihood

$$\log \mathcal{L}(\theta|\Delta S_1, \ldots, \Delta S_n) = \sum_{i=1}^{n} f_{\Delta S_i}(\Delta S_i|\theta). \tag{33}$$

Differential evolution (DE) is a search heuristic introduced by Storn and Price (1997) and is a genetic evolutionary algorithm that uses biology-inspired operations of crossover, mutation, and selection on a population in order to minimize

[10] We used both `nlminb` in R and `fminsearch` in Matlab.

an objective function over the course of successive generations. It is an efficient algorithm that is well suited for optimization problems with functions that exhibit many optima and discontinuities, since it's a heuristic method and does not need to evaluate the gradient of the function at every iteration[11]. For further details on the methodology we refer to the original paper that introduced the method. We apply the DE algorithm to minimize the log-likelihood (33) since using DE to minimize (32) increases drastically the computation time. Both methods work well for our purposes and produce similar estimates. We estimate (30) on each firm's log-returns for a GJD and on each firm's specific residuals for a DEJD. Table 1 and Table 3 show the mean parameter estimates across firms for both sets of estimations.

4.2 Stage Two Estimation

In the second stage we estimate the following investment equations:

$$\begin{aligned} \frac{I_{it}}{K_{it-1}} &= \beta_1 \Delta Y_{it} + \beta_2 \frac{CF_{it}}{K_{it-1}} + \beta_3 \frac{CF_{it-1}}{K_{it-2}} + \beta_4 Size_{it} + \beta_5 q_{it} + \\ &\quad \beta_6 \left(\sigma_i \times \Delta Y_{it}\right) + + \beta_7 \left(\lambda_i \times \Delta Y_{it}\right) + \alpha_i + \tau_t + \epsilon_{it} \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{I_{it}}{K_{it-1}} &= \beta_0 \frac{I_{it-1}}{K_{it-2}} + \beta_1 \Delta Y_{it} + \beta_2 \frac{CF_{it}}{K_{it-1}} + \beta_3 \frac{CF_{it-1}}{K_{it-2}} + \beta_4 Size_{it} + \\ &\quad \beta_5 q_{it} + \beta_6 \left(\sigma_i \times \Delta Y_{it}\right) + \beta_7 (\lambda_i^d \times \Delta Y_{it}) + \beta_8 \left(\lambda_i^u \times \Delta Y_{it}\right) + \\ &\quad \alpha_i + \tau_t + \epsilon_{it}. \end{aligned} \tag{35}$$

In both equations I_{it} is the investment of firm i in period t and K_{it-1} is the measure of capital stock at the end of the previous period. Following Cooper and Haltiwanger (2006) and Caballero et al. (1995) we define the investment measure as:

$$I_{it} = CAPEX_{it} - DISP_{it}$$

where I_{it} is investment, $CAPEX_{it}$ represents capital expenditure, i.e. the funds used to acquire fixed assets other than those associated with acquisitions, and $DISP_{it}$ is the disposal or retirement of fixed assets. Our focus is therefore on the investment rate (I_{it}/K_{it-1}) which can be positive or negative. The reason we adopt this modified definition of investment is due to the importance of both positive and negative capital stock adjustment in our model. The intuitive gist of how we map the data to our model dynamics is the following: every time the state variable s_t hits the upper threshold s_I the firm will activate a burst of investment dK^+. Similarly, every time the lower threshold s_D is hit, disinvestment dK^- will be activated. The net investment rate at time t, which will be our outcome variable in equations (34) and (35), will then be the sum of the investment bursts, minus disinvestment.

[11]This method has already been applied to Gaussian jump diffusions by Ardia et al. (2011)

Our econometric specification controls for investment opportunities by including a measure of *average q* (q_{it}), as defined by Hayashi (1982), real sales growth (ΔY_{it}), firm size and cash flow (CF) since these variables are informative of a firm's investment decisions. The Q model of investment relates firm investment rate to its *marginal* q, which is reflected in the present discounted value of expected future profits. Empirically, however, we can only observe the *average q*, i.e.Tobin's q whose low explanatory power, mainly attributed to measurement error, is well documented in the literature. We therefore include two other control variables, real sales growth and cash flow that provide information on firm investment behaviour and may predict marginal q. Both these terms provide additional information by capturing expectations of profitability at longer horizons, or reflect misspecification of the basic Q model; furthermore, cash flow may specifically reflect financing constraints or capital market imperfections (Bond et al. (2004); Bulan (2005); Fazzari et al. (1988)). Additionally we include firm size as a control, since a large literature shows that size affects the firm's access to external capital markets and hence may influence it's investment rate (Kadapakkam et al. (1998); Audretsch and Elston (2002);Whited (1992)).Lastly, our second specification is dynamic and includes lagged investment rate, the influence of which on current investment is a well documented result in the investment literature (Gilchrist and Himmelberg (1995); Gilchrist et al. (2014); Eberly et al. (2012)). Equation (35) includes interaction terms between real sales growth and our uncertainty components (σ_i, λ_i), which are discussed in detail in the next section. Finally we also control for firm specific fixed effects (α_i) and firm-invariant time-specific effects (τ_t). With these controls in place and exploiting the panel nature of our data we estimate (34) and (35).

4.2.1 Baseline Estimation: Gaussian Jump Diffusion

In the first stage, for our baseline estimation we fit a Gaussian distribution to the random jump magnitude on the firms' daily stock returns, which yields five parameters per firm: ($\lambda_i, \sigma_i, \mu_i, \mu_{J_i}, \sigma_{J_i}$). The mean and standard deviation across firms of the estimated parameters are given in Table 1 for both MMLE and DE estimation techniques. For this set of parameters we then estimate (34) using the MMLE results[12].

Table 1: Mean parameter estimates across firms, Gaussian jump diffusion

	μ	σ	λ	μ_J	σ_J
(MMLE)	-0.002	0.285	2.780	-0.004	0.224
	(0.222	(0.145)	(2.217)	(0.167)	(0.172)
(DE)	-0.046	0.295	3.452	-0.043	0.168
	(0.245)	(0.147)	(2.216)	(0.280)	(0.091)

[12]DE results yield similar second-stage estimations, which we omit for brevity.

Volatility and drift of a firm are represented by σ and μ. The jump intensity is represented by λ and the jump sizes are normally distributed with parameters μ_J and σ_J. The interaction terms in our investment equations are of primary interest to us as they exhibit the effect of uncertainty on the impact effect of demand shocks on investment decision. The first term, widely used in the literature, is an interaction between the volatility component of uncertainty and sales growth ($\sigma_i \times \Delta Y_{it}$). The second term in equation (34) is an interaction between the jump intensity of a firm and sales growth ($\lambda_i \times \Delta Y_{it}$). Although this distribution does not explicitly give us parameters to differentiate between an up and down jump, it provides us with a good starting point to test how jump frequency can influence the firm investment rate in the presence of a demand shock. Since our model predicts that an increase in frequency of jumps, for a Gaussian jump diffusion, increases the firm's inaction region, this interaction term will be indicative of if and how jumps, irrespective of their direction, impact the firm's investment rate.

Table 2: Investment, Jumps & Volatility

Dep.var.: I_{it}/K_{it-1}	(1) Baseline	(2) GJD	(3) DEJD	(4) DEJD
Sales growth (ΔY_{it})	0.145***	0.178*	0.327***	0.781***
	(0.0464)	(0.0914)	(0.125)	(0.226)
Cash Flow ($\frac{C_{it}}{K_{it-1}}$)	0.0235	0.00956	0.0117	-0.0631
	(0.0306)	(0.0244)	(0.0445)	(0.0601)
Lagged cash flow ($\frac{C_{it-1}}{K_{it-2}}$)	0.0642**	0.0774***	0.0765***	0.0892*
	(0.0270)	(0.0237)	(0.0247)	(0.0480)
$Size_{it}$	0.0659	0.0523	0.0496	-0.0388
	(0.0427)	(0.0336)	(0.0409)	(0.0278)
Tobin's q (q_{it})	0.0227*	0.0240**	0.0253**	0.0747**
	(0.0118)	(0.0120)	(0.0129)	(0.0312)
Lagged Investment				0.233***
(I_{it-1}/K_{it-2})				(0.0800)
Volatility $\times$ sales growth		-0.0649*	-0.148**	-0.350***
($\sigma_i \times \Delta Y_{it}$)		(0.0394)	(0.0663)	(0.108)
Jump freq. $\times$ sales growth		-0.0366*		
($\lambda_i \times \Delta Y_{it}$)		(0.0204)		
Negative jump freq. $\times$ sales growth			-0.0586**	-0.194**
($\lambda_i^d \times \Delta Y_{it}$)			(0.0290)	(0.0863)
Positive jump freq. $\times$ sales growth			-0.0141	0.0191
($\lambda_i^u \times \Delta Y_{it}$)			(0.0295)	(0.0523)
N	3194	3194	3194	2385
Hansen ($p-value$)				0.292
Second-order serial correlation ($p-value$)				0.964
Third-order serial correlation ($p-value$)				0.169

Note: Sample period is 2003-2017 at an annual frequency (No. firms = 403). Dependent variable in logs. Bootstrapped standard errors clustered at the firm level shown in parenthesis. Time dummies are included (but not reported) in all specifications. * $p < .1$, ** $p < .05$, *** $p < .01$

Table 2 shows the estimation results. In column (1) we report results for a simple investment regression, without uncertainty, to establish the relationship between our control variables and firm investment rate. We find that the coefficients on all the independent variables are correctly signed and significant with the exception of current cash flow and size. In column (2) we introduce the interaction terms which gives us an insight into how our components of uncertainty influence the firm's investment rate. Our result of interest here is the significant, albeit weakly, negative coefficient on the interaction term between real sales growth and jump intensity - this encourages us to explore this relation further as it is in line with our model's implication that an increase in the frequency of jumps may increase the inaction region by affecting its investment and/or disinvestment threshold.

4.2.2 Main Estimation: Double Exponential Jump Diffusion

For our main estimation we fit a double exponential jump diffusion on the residuals, giving us six parameters per firm: $\left(\lambda_i^d, \lambda_i^u, \sigma_i, \mu_i, \eta_{1i}, \eta_{2i}\right)$. The mean and standard deviation across firms of the estimated parameters are given in Table 3 for both MMLE and DE estimation techniques. For consistency with the previous section, we use the MMLE parameter set for the second-stage estimation[13]. The jump asymmetry modeled by DEJD allows us to distinctly characterize firm investment behaviour for negative and positive jumps. We therefore estimate (35) for this set of parameters.

Table 3: Mean parameter estimates across firms, double exponential jump diffusion

	μ	σ	η_1	η_2	λ^u	λ^d
(MMLE)	0.003	0.054	4.0832	4.138	3.619	3.707
	(0.052)	(0.017)	(1.234)	(1.267)	(0.568)	(0.508)
(DE)	−0.002	0.039	3.560	3.640	3.963	4.034
	(0.009)	(0.017)	(1.614)	(1.552)	(0.587)	(0.570)

Firm volatility is represented by σ and negative and positive jump frequencies are represented by λ^d and λ^u. The mean positive jump size is given by $1/\eta_1$ and for negative jumps by $1/\eta_2$. In (35) we have three interaction terms of significance, in addition to volatility $(\sigma_i \times \Delta Y_{it})$ we separately observe the responsiveness of firm investment to demand for negative jumps $(\lambda_i^d \times \Delta Y_{it})$ and positive jumps $(\lambda_i^u \times \Delta Y_{it})$. Table 2 column (3) shows the estimation results without lagged investment and column (4) is the full dynamic specification. We estimate this using system GMM (Arellano and Bover (1995); Blundell and Bond (1998)) which combines a system of equations in first differences using suitably lagged levels of

[13] Second-stage estimates with DE parameters yields similar results and is therefore again omitted.

endogenous variables as instruments (similar to Arellano-Bond first differenced estimator), with equations in levels for which lagged differences of endogenous variables are used as instruments. Unobserved firm specific effects are eliminated from the first-differenced transformation. We see that the firm fundamental variables are correctly signed and significant. More importantly, we find that the volatility and negative jump frequency interaction terms are negative and significant thus indicating that the firm investment response to demand shocks is indeed lower, not only at higher levels of volatility but also higher frequency of negative jumps. This validates the result that not only the frequency but the direction of jumps play a role. If the frequency of negative jumps is large enough, the increase in the inaction area will be significant. Unsurprisingly, we find the positive jump frequency interaction term to be insignificant. This is consistent with our model which states that positive jumps have a significant negative impact on the (dis)investment threshold at very high levels of volatility and positive impact at low levels of volatility. The insignificance of this interaction term could imply that the underlying volatility for our panel of firms is not high or low enough to capture the effect of the positive jumps effect.

In Table 4, column (1) we split our sample into terciles, based on the firm's growth rate and compare the lowest growing firms to the highest ones. Our model states that firms with low growth would be at a much higher disadvantage than the high growing firms. The uncertainty interaction terms for the low growing firms are negative and significant, thus confirming that these firms are more sensitive to *negative* jumps and volatility than the high growth firms. The second column focuses on liquidity and financial constraints, which play a fundamental role in influencing firm behaviour. There is a voluminous literature studying how a firm's investment decision is influenced by the financing frictions it faces[14]. If a firm is financially constrained they are more prone to the effects of uncertainty, thus magnifying their sensitivity to investment decisions. In column (2), we examine the robustness of the jumps, volatility and investment relationship by splitting our sample based on a popular measure of financial constraint - the Kaplan and Zingales (KZ) index[15] which combines several accounting and market firm characteristics. A higher index value suggests a firm is more constrained. The five variables, along with the signs of their coefficients in the KZ index, are: cash flow (negative), the market to book ratio (positive), leverage (positive), dividends (negative), and cash holdings (negative). For the KZ index we classify as constrained firms those ranked in the top tercile while those in the bottom tercile are classified as unconstrained. In column (2), we see that negative jumps have a significant negative impact on the response of investment to demand shocks for constrained firms, while the effect for unconstrained firms is only weakly significant. We also find both the volatility interaction terms to be significant and negative.

[14]See Fazzari et al. (1988); Whited (1992); Froot et al. (1993); Kaplan and Zingales (1995); Gomes (2001); Denis and Sibilkov (2009)

[15]Kaplan and Zingales (1995); Lamont et al. (2001)

Table 4: Firm Growth & Financial Constraint

Dep.var.: I_{it}/K_{it-1}	(1) Firm Growth	(2) Financial Constraint
Lagged Investment	0.178^{**}	0.221^{**}
(I_{it-1}/K_{it-2})	(0.0792)	(0.0880)
Cash Flow ($\frac{C_{it}}{K_{it-1}}$)	-0.0785^{*}	-0.0721
	(0.0421)	(0.0461)
Lagged cash flow ($\frac{C_{it-1}}{K_{it-2}}$)	0.0799^{**}	0.113^{**}
	(0.0405)	(0.0450)
$Size_{it}$	-0.0260	-0.0414
	(0.0291)	(0.0312)
Tobin's q (q_{it})	0.0757^{**}	0.0720^{**}
	(0.0371)	(0.0320)
Sales growth (ΔY_{it})	0.840^{***}	0.989^{***}
	(0.220)	(0.222)
Firm growth(low)×Volatility × sales growth	-0.390^{***}	
$(growth^l_{it} \times \sigma_i \times \Delta Y_{it})$	(0.136)	
Firm growth(high)×Volatility × sales growth	-0.304^{***}	
$(growth^h_{it} \times \sigma_i \times \Delta Y_{it})$	(0.111)	
Firm growth(low)×Negative jump freq. × sales growth	-0.307^{**}	
$(growth^l_{it} \times \lambda^d_i \times \Delta Y_{it})$	(0.141)	
Firm growth(high)×Negative jump freq. × sales growth	-0.154^{*}	
$(growth^h_{it} \times \lambda^d_i \times \Delta Y_{it})$	(0.0929)	
Firm growth(low)×Positive jump freq. × sales growth	0.119	
$growth^l_{it} \times \lambda^u_i \times \Delta Y_{it})$	(0.127)	
Firm growth(high)×Positive jump freq. × sales growth	-0.0430	
$(growth^h_{it} \times \lambda^u_i \times \Delta Y_{it})$	(0.0622)	
unconstrained × Volatility × sales growth		-0.410^{***}
$(fin.const^u_{it} \times \sigma_i \times \Delta Y_{it})$		(0.123)
constrained × Volatility × sales growth		-0.446^{***}
$(fin.const^c_{it} \times \sigma_i \times \Delta Y_{it}$		(0.112)
unconstrained × Negative jump freq. × sales growth		-0.200^{*}
$(fin.const^u_{it} \times \lambda^d_i \times \Delta Y_{it})$		(0.109)
constrained × Negative jump freq. × sales growth		-0.340^{***}
$(fin.const^c_{it} \times \lambda^d_i \times \Delta Y_{it})$		(0.113)
unconstrained × Positive jump freq. × sales growth		-0.0301
$(fin.const^u_{it} \times \lambda^u_i \times \Delta Y_{it})$		(0.0686)
constrained × Positive jump freq. × sales growth		0.120
$(fin.const^c_{it} \times \lambda^u_i \times \Delta Y_{it})$		(0.0888)
N	1536	2347
Hansen ($p - value$)	0.186	0.326
Second-order serial correlation ($p - value$)	0.947	0.836
Third-order serial correlation ($p - value$)	0.182	0.221

Note: Sample period is 2003-2017 at an annual frequency (No. firms = 403). Dependent variable in logs. Time dummies are included (but not reported) in all specifications. * $p < .1$, ** $p < .05$, *** $p < .01$

5 Conclusions

In this paper we investigate the effect of different sources of uncertainty on firm investment dynamics. We present a model of optimal investment and disinvestment with partial irreversibility for a firm that faces stochastic business conditions driven by a jump diffusion, allowing us to incorporate discontinuous uncertainty related to information arrival, sudden changes and large shocks. This framework allows us to study the effect of uncertainty on the investment and disinvestment thresholds, and we discover that the presence of jumps can drastically increase the area in which it is not profitable for a firm to invest. We discover that the investment rate is reduced by an increase in the frequency of negative jumps, and that firms disinvest quicker in order to protect themselves from the possibility of a large negative jump that would cause shutdown of operations. We also find that positive jumps can have an ambiguous effect depending on the underlying volatility. We show that firms with lower growth are more affected by discontinuous uncertainty than firms with medium to high growth. These results imply that the standard Gaussian framework consistently underestimates the negative effect of uncertainty on firm investment decisions. We test the model implications on a panel of UK firm-level data dating from 2003 to 2018: we first estimate the jump diffusion parameters for each firm for both Gaussian and Double Exponential jump diffusions by means of multinomial maximum likelihood and differential evolution techniques, and subsequently we estimate investment equations on our panel of firms. We show that our empirical findings are in line with the theoretical implications of the model.

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A Derivation of the profit function

Assume each firm has a Cobb-Douglas production function with decreasing returns to scale of the form

$$Q_t = \omega_t K_t^{\alpha} L_t^{\beta}$$

where we have productivity ω_t, capital K and labor L. The firm faces an isoelastic demand curve with elasticity (ϵ)

$$Q_t = B_t P_t^{-\epsilon}$$

where B is a demand shifter. These can be combined into a revenue function

$$R_t(\omega, B, K, L) = B_t^{\frac{1}{\epsilon}} \omega_t^{1-\frac{1}{\epsilon}} K_t^{\alpha(1-\frac{1}{\epsilon})} L_t^{\beta(1-\frac{1}{\epsilon})}$$

Now let us denote $a = \alpha\left(1 - \frac{1}{\epsilon}\right)$ and $b = \beta\left(1 - \frac{1}{\epsilon}\right)$: we obtain

$$X_t^{1-a-b} = B_t^{\frac{1}{\epsilon}} \omega_t^{1-\frac{1}{\epsilon}}$$

where X now combines the firm demand and productivity. We assume labour to be a flexible factor of production that can be instantaneously adjusted, and thus the firm chooses L to maximize their instantaneous profit at every instant t:

$$\begin{aligned}
\pi(X, K) &= \max_{L_t}(X_t^{1-a-b} K_t^a L_t^b - wL_t) \\
&= \left(\frac{b}{w}\right)^{\frac{b}{1-b}} X_t^{1-\frac{a}{1-b}} K_t^{\frac{a}{1-b}} \\
&= h X_t^{\gamma} K_t^{1-\gamma}
\end{aligned}$$

B Proof of Proposition 1

Using the Ito-Levy formula for a geometric Lévy process we have

$$\begin{aligned}
d\ln(X_t) &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \lambda \int_{\Omega} \Big[\ln[X_t(1+z)] - \ln(X_t) - X_t^{-1} z X_t\Big] F(dz) + \\
&+ \sigma dW_t + \Big[\ln[X_t(1+z)] - \ln(X_t)\Big](dN_t - \lambda dt)
\end{aligned}$$

which is the sum of the geometric drift, the expectation of the jump magnitudes and the sum of two martingales. The dynamics of the capital stock when investment is zero are given by $d\log(K_t) = -\delta dt$, therefore

$$\begin{aligned}
d\ln\left(\frac{X_t}{K_t}\right) &= \left(\mu + \delta - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + \\
&+ \lambda \int_{\Omega} \Big[\ln[X_t(1+z)] - \ln(X_t) - X_t^{-1} z X_t\Big] F(dz) + \\
&+ \Big[\ln[X_t(1+z)] - \ln(X_t)\Big](dN_t - \lambda dt) \qquad (36)
\end{aligned}$$

We can now consider the variable $s_t = X_t/K_t$, as demand scaled by capital (firm size). Adding and subtracting $\log(K_t)$ and dividing X_t by K_t in the jump parts of (36) one can immediately recognize that s_t evolves according to the geometric Lévy process

$$ds_t = (\mu + \delta)\, s_t dt + \sigma s_t dW_t + s_{t^-} \int_\Omega z\tilde{N}(dt, dz)dt \tag{37}$$

C Proof of Proposition 2

The solution of the homogeneous part is:

$$q_H(s_t) = \tilde{A}s_t^\gamma - B \rightarrow V_H(X_t, K_t) = AX_t^\gamma K_t^{1-\gamma} - BK_t$$

which implies a general solution of the form

$$\begin{aligned} q(s_t) &= \tilde{A}s_t^\gamma - B + \tilde{C}_1 s_t^{\xi_1} + \tilde{C}_2 s_t^{\xi_2} \\ &\downarrow \\ V(X_t, K_t) &= AX_t^\gamma K_t^{1-\gamma} - BK_t + C_1 X_t^{\xi_1} K_t^{1-\xi_1} + C_2 X_t^{\xi_2} K_t^{1-\xi_2} \end{aligned}$$

For the homogeneous part, substituting the guess yields

$$q_H(s_t) = \tilde{A}s_t^\gamma - B \rightarrow V_H(X_t, K_t) = AX_t^\gamma K_t^{1-\gamma} - BK_t$$

which implies a general solution of the form

$$\begin{aligned} q(s_t) &= \tilde{A}s_t^\gamma - B + \tilde{C}_1 s_t^{\xi_1} + \tilde{C}_2 s_t^{\xi_2} \\ &\downarrow \\ V(X_t, K_t) &= AX_t^\gamma K_t^{1-\gamma} - BK_t + C_1 X_t^{\xi_1} K_t^{1-\xi_1} + C_2 X_t^{\xi_2} K_t^{1-\xi_2} \end{aligned}$$

For the homogeneous part, substituting the guess yields

$$\begin{aligned} \tilde{A} &= \frac{(1-\gamma)a}{\rho + \delta - \lambda\phi(z) - \gamma\left(\mu + \delta - \frac{\sigma^2}{2}\right) - \gamma^2 \frac{\sigma^2}{2}} \\ B &= \frac{M}{\rho + \delta} \end{aligned}$$

where the jump parts are given by the probability integrals

$$\begin{aligned} \phi(z) &= \int_\Omega \Big[(1+z)^\gamma - 1 - \gamma z\Big] F(dz) \\ \phi_1(z) &= \gamma \int_\Omega zF(dz) \\ \phi_2(z) &= \int_\Omega [1 - (1+z)^\gamma] F(dz) \end{aligned}$$

Note that since $\gamma < 1$, the fact that the integral is bounded below at -1 guarantees real solutions. The complementary solution solves

$$\left[\rho + \delta + \lambda \int_\Omega \Big(1 - (1+z)^\gamma\Big) F(dz)\right] q(s_t) =$$
$$s_t \left(\mu + \delta - \lambda\gamma \int_\Omega zF(dz)\right) q'(s_t) + s_t^2 \frac{\sigma^2}{2} q''(s_t)$$

Trying a solution of the form $q(s) = s^\xi$ allows to obtain the two roots

$$\begin{aligned} \xi_{2,1} &= \frac{1}{\sigma^2}\left[\left(\frac{\sigma^2}{2} - \mu - \delta + \lambda\phi_1(z)\right) \pm \right. \\ &\left. \pm\sqrt{\left(\mu + \delta - \lambda\phi_1(z) - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\left[\rho + \delta + \lambda\phi_2(z)\right]}\right] \end{aligned} \tag{38}$$

where we have $\phi_1(z) = \gamma \int_\Omega zF(dz)$ and $\phi_2(z) = \int_\Omega [1 - (1+z)^\gamma] F(dz)$. y means of the superposition principle we obtain the solution for the marginal value of capital

$$q(s_t) = \tilde{A} s_t^\gamma - B + C_1 s_t^{\xi_1} + C_2 s_t^{\xi_2}. \tag{39}$$

In (39), $\tilde{A} s_t^\gamma$ is the value of the assets in place and $C_1 s_t^{\xi_1} + C_2 s_t^{\xi_2}$ is the value of the investment options, the first for disinvestment and the second for investment.

Integrating in K yields the form of the solution for the value function

$$V(X_t, K_t) = A X_t^\gamma K_t^{1-\gamma} - B K_t + C_1 X_t^{\xi_1} K_t^{1-\xi_1} + C_2 X_t^{\xi_2} K_t^{1-\xi_2} + C_K$$

where the constants $\tilde{A}, B$ are given (40) and (41). Using the natural boundary condition $V(0,0) = 0$ one easily obtains $C_K = 0$. Note that coherently with what shown before, it is homogeneous of degree one, i.e. $V(s) = V(X,K)/K$.

$$\tilde{A} = \frac{(1-\gamma)a}{\rho + \delta - \lambda\phi(z) - \gamma\left(\mu + \delta - \frac{\sigma^2}{2}\right) - \gamma^2 \frac{\sigma^2}{2}} \tag{40}$$

$$B = \frac{M}{\rho + \delta} \tag{41}$$

where the jump part is given by the probability integral

$$\phi(z) = \int_\Omega \Big[(1+z)^\gamma - 1 - \gamma z\Big] F(dz).$$

Note that since $\gamma < 1$, the fact that the integral is bounded below at -1 guarantees real solutions. The complementary solution solves

$$\left[\rho+\delta+\lambda\int_\Omega\left(1-(1+z)^\gamma\right)F(dz)\right]q(s_t)=$$
$$s_t\left(\mu+\delta-\lambda\gamma\int_\Omega zF(dz)\right)q'(s_t)+s_t^2\frac{\sigma^2}{2}q''(s_t)$$

Trying a solution of the form $q(s)=s^\xi$ gives

$$\left[\rho+\delta+\lambda\int_\Omega\left(1-(1+z)^\gamma\right)F(dz)\right]s^\xi$$
$$=\xi\left(\mu+\delta-\lambda\gamma\int_\Omega zF(dz)\right)s^\xi+\xi(\xi-1)\frac{\sigma^2}{2}s^\xi$$

which allows to obtain the two roots

$$\begin{aligned}\xi_{2,1} &= \frac{1}{\sigma^2}\left[\left(\frac{\sigma^2}{2}-\mu-\delta+\lambda\phi_1(z)\right)\pm\right. \\ &\left.\pm\sqrt{\left(\mu+\delta-\lambda\phi_1(z)-\frac{\sigma^2}{2}\right)^2+2\sigma^2\left[\rho+\delta+\lambda\phi_2(z)\right]}\right]\end{aligned}\tag{42}$$

where we have $\phi_1(z)=\gamma\int_\Omega zF(dz)$ and $\phi_2(z)=\int_\Omega[1-(1+z)^\gamma]F(dz)$. In (39), $\tilde{A}s_t^\gamma$ is the value of the assets in place and $C_1s_t^{\xi_1}+C_2s_t^{\xi_2}$ is the value of the investment options, the first for disinvestment and the second for investment.

Integrating in K yields the form of the solution for the value function

$$V(X_t,K_t)=AX_t^\gamma K_t^{1-\gamma}-BK_t+C_1X_t^{\xi_1}K_t^{1-\xi_1}+C_2X_t^{\xi_2}K_t^{1-\xi_2}+C_K$$

and using a natural boundary condition $V(0,0)=0$ we immediately obtain $C_K=0$, which satisfies homogeneity as required.

D Data

- $CAPEX_{it}$ represents the capital expenditures of a firm i.e. the funds used to acquire fixed assets other than those associated with acquisitions. It includes additions to property, plant and equipment and investments in machinery and equipment. Datastream item WC04601
- $DISP_{it}$ represents disposal of fixed assets i.e. the amount a company received from the sale of property, plant and equipment. Datastream item WC04351

$$I_{it}=CAPEXP_{it}-DISP_{it}$$

- K_{it} represents the firm capital stock taken as the net property and plant equipment. Datastream item WC02501.
- Y_{it} represents net sales i.e. gross sales and other operating revenue less discounts, returns and allowances. Datastream item WC01001.
- C_{it} represents cash flow i.e. funds from operations - the sum of net income and all non-cash charges or credits. Datastream item WC04201.
- q_{it} represents Tobin's Q i.e. market value of assets by replacement value of assets which can be calculated with market capitalization (WC08001), total liabilities (WC03351) and common equity (WC03501)
- $Size_{it}$: total assets representing the sum of total current assets, long term receivables, investment in unconsolidated subsidiaries, other investments, net property plant and equipment and other assets. Datastream item WC02999.
- Kaplan-Zingales Index: $-1.002 \times \frac{\text{cash flow}_{it}}{\text{property, plant and equipment (net)}_{it-1}} + 0.283 \times q_{it} + 3.139 \times \frac{\text{total debt}_{it}}{\text{total capital}_{it}} - 39.368 \times \frac{\text{dividends paid}_{it}}{\text{property, plant and equipment (net)}_{it-1}} - 1.315 \times \frac{\text{cash holdings}_{it}}{\text{property, plant and equipment (net)}_{it-1}}$
  - Total debt represents all interest bearing and capitalized lease obligations. It is the sum of long and short term debt. Datastream item WC03255
  - Total capital represents the total investment in the company. It is the sum of common equity, preferred stock, minority interest, long-term debt, non-equity reserves and deferred tax liability in untaxed reserves. For insurance companies policyholders' equity is also included. Datastream item WC03998
  - Dividends paid represent the total common and preferred dividends paid to shareholders of the company. Datastream item WC04551
  - Cash holdings represents the sum of cash and short term investments. Datastream item WC02001

All financial variables have been deflated by the GDP deflator (base year: 2010)